Improper Integrals

63. Determine the domain of the function $f(x) = \frac{1}{\sqrt[3]{(x-1)^2}}$. Then compute the following integral and explain its geometric meaning:

$$\int_{-1}^{2} \frac{\mathrm{d}x}{\sqrt[3]{(x-1)^2}}.$$

Sequence of Functions and Uniform Convergence

64. The sequence of functions $(f_n)_{n\in\mathbb{N}}$ is defined by:

$$f_n : [0,1] \to \mathbb{R}, \qquad f_n(x) = 2x + \frac{x}{n}.$$

(a) Determine the limit function $f(x) = \lim_{n\to\infty} f_n(x)$.

(b) Is the limit function $f(x)$ continuous on $[0,1]$? Provide a detailed explanation.

(c) Verify whether $(\lim_{n\to\infty} f_n(x))' = \lim_{n\to\infty} f_n'(x)$ holds for all $x \in [0,1]$.

(d) Check if $\int_0^1 \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x$.

65. The sequence of functions $f_n : [0,\infty) \to \mathbb{R}$ is given by $f_n(x) = \frac{nx}{1+nx}$. Determine the limit function and establish whether the sequence converges uniformly on $[a,\infty)$, where $a > 0$. Does the sequence converge uniformly on $[0,\infty)$?

66. The sequence of functions $(f_n)_{n\in\mathbb{N}}$ is defined by:

$$f_n : [0,1] \to \mathbb{R}, \qquad f_n(x) = \frac{1}{2}x^n.$$

(a) To which function does the sequence $(f_n)_{n\in\mathbb{N}}$ converge on $[0,1]$?

(b) Is the convergence in (a) uniform?

All above math problems are taken from the following website:
`https://osebje.famnit.upr.si/~penjic/teaching.html`.
THE READER CAN FIND ALL SOLUTIONS TO THE GIVEN PROBLEMS ON THE SAME PAGE.